International Journal of Theoretical Physics, Vol. 43, No. 6, June 2004 (© 2004)

Decay Law of Moving Unstable Particle

Mihail Shirokov[1]

Received

Quantum relativistic decay law of moving unstable particle is analytically calculated in the model case of the Breit–Wigner mass distribution. It turns out that Einstein time dilation of the moving particle decay holds approximately at times when the decay is exponential. The related correction is calculated analytically. Being very small at these times it is practically unobservable. It is shown that Einstein dilation fails for large times t when decay is not exponential. An unstable system of the kind of K_0-meson (which is the superposition of K_s and K_l) is also considered. In this case, the violation of Einstein dilation is shown to be appreciable at all times under some condition

KEY WORDS: special relativity; relativistic quantum mechanics; unstable particles; decay law; time dilation; K_0-meson decay.

1. INTRODUCTION

Experimenters showed that the lifetime τ of unstable particles moving with velocity v is equal to $\tau_0\gamma$, where τ_0 is the lifetime of the particle at rest and $\gamma = (1 - v^2/c^2)^{-1/2}$. Usual explanation of the fact is based on the special theory of relativity. For example, Møller (1972) sets forth it as follows:

> In view of the fact that an arbitrary physical system can be used as a clock, we see that any physical system which is moving relative to a system of inertia must have a slower course of development than the same system at rest. Consider for instance a radioactive process. The mean life τ of the radioactive substance, when moving with a velocity v, will thus be larger than the mean life τ_0 when the substance is at rest. From (2.36) we obtain immediately $\gamma = (1 - v^2/c^2)^{-1/2}\tau_0$.

This argumentation may be complemented by the following possible definition of the unit of time which radioactive substance provides: this is the time interval during which the amount of the substance decreases twice, e.g.

However, the standard clocks of the relativity theory are used when obtaining Eq. (2.36)

$$\Delta t = t_2 - t_1 = \gamma(t_2' - t_1') = (1 - v^2/c^2)^{-1/2}\Delta\tau$$

[1] Joint Institute for Nuclear Research, Moscow, Russia; e-mail: shirokov@thsun1.jinr.ur.

0020-7748/04/0600-1541/0 © 2004 Springer Science+Business Media, Inc.

which Møller mentions. He begins the derivation of this equation with the phrase:

> Consider a standard clock C' which is placed at rest in S' at a point on the x'-axis with the coordinate $x' = x'_1$.

However, such a quantum clock as an unstable particle cannot be at rest (i.e., cannot have zero velocity or zero momentum) and simultaneously be at a definite point (due to the quantum uncertainty relation). So, the standard derivation of the moving clock dilation is inapplicable for the quantum clock. The related quantum-mechanical derivation must contain some reservations and corrections. Here another way of deriving the time dilation is used: to find relativistic quantum decay law of a moving unstable particle (momentum $p \neq 0$) and to compare it with the decay law of the particle at rest. Both the laws refer to one Lorentz frame (e.g., laboratory frame). Lorentz transformations of the space–time coordinates are not needed as well as the space coordinates themselves.

This approach to the derivation of the time dilation was used by Exner (1983) and Stefanovich (1996).

A simple derivation of the probability amplitude $\mathcal{A}_p(t)$ of the decay (more exactly nondecay) of the particle with momentum $\mathbf{p}$ is suggested in Section 2. Distinctions from Exner and Stefanovich derivations are discussed.

Let us define the terminology used below. The equation $\tau_p = \tau_0\gamma$ is equivalent to the the equation

$$|\mathcal{A}_p(t)|^2 = |\mathcal{A}_0(t/\gamma)|^2 \tag{1}$$

if $|\mathcal{A}_p|^2$ and $|\mathcal{A}_0|^2$ depend upon t as

$$|\mathcal{A}_p(t)|^2 \sim \exp(-t/\tau_p), \quad |\mathcal{A}_0(t)|^2 \sim \exp(-t/\tau_0).$$

Equation (1) or the substitution $t \to t/\tau$ used in Eq. (1) will be called Einstein dilation ED (in this respect see also Eq. (2.37) in Møller, 1972).

Calculations of quantum probabilities $|\mathcal{A}_p(t)|^2$ and $|\mathcal{A}_0(t)|^2$ show that ED does not hold exactly. Numerical calculations were performed by Stefanovich (1996). Here analytical evaluation of $\mathcal{A}_p(t)$ is carried out, see Section 3 and Appendix. It allows us to determine the region of times t, where ED is approximately valid, and its accuracy. It also shows that ED fails under some conditions. In particular, I shall consider in Section 4 a variant of the quantum clock for which appreciable deviations from ED take place at all times.

For conclusion see Section 5. The analytical evaluation of $\mathcal{A}_p(t)$ is presented in Appendix.

2. NONDECAY LAW OF MOVING UNSTABLE PARTICLE

Let us consider a relativistic theory which describes unstable particles, products of their decay, and the corresponding interactions. A field theory may be an example. Such a theory must contain operators of total energy and momentum $\hat{H}$, $\hat{\mathbf{P}}$ (the generators of time and space translations), total angular momentum, and generators of Lorentz boosts.

Suppose that at the initial moment $t = 0$ there is one unstable system (particle) having definite momentum p'. As there are no other particles p' is an eigenvalue of the total momentum $\hat{\mathbf{P}}$. The state vector $\Psi_{p'}$ of this system is the related $\hat{\mathbf{P}}$ eigenvector which may be represented in the momentum representation as

$$\langle \vec{p} | \Psi_{\vec{p}'} \rangle = \delta_{\vec{p},\vec{p}'} \psi_0. \tag{2}$$

Here ψ_0 describes the unstable system in its rest frame. For example, ψ_0 may describe an excited state of the hydrogen atom in the atom rest frame. In Eq. (2), $\delta_{\vec{p},\vec{p}'}$ is the Kronecker symbol: I suppose that $\hat{\mathbf{P}}$ has a discrete spectrum (the 2 system is in a large space volume and usual periodicity conditions are imposed or the volume opposite boundaries are identified). Thus, $\Psi_{p'}$ has the unit norm.

Consider a common eigenvector $\Phi_{p'\mu}$ of $\hat{\mathbf{P}}$ and $\hat{H}$. Similarly to Eq. (2) one has

$$\langle \vec{p} | \Phi_{\vec{p}'\mu} \rangle = \delta_{\vec{p},\vec{p}'} \phi_{0\mu}. \tag{3}$$

Here $\phi_{0\mu}$ is the common $\hat{H}$ and $\hat{\mathbf{P}}$ eigenvector corresponding to the zero eigenvalue of $\hat{\mathbf{P}}$. The related $\hat{H}$ eigenvalue may be called the mass μ. Let us expand $\Psi_{p'}$ in vectors $\Psi_{\vec{p}'\mu}$

$$\Psi_{p'} = S_\mu c(\mu) \Psi_{\vec{p}'\mu}. \tag{4}$$

Here S_μ denotes sum and/or integral over μ (it will be refined below). I suppose that $c(\mu)$ in Eq. (4) does not depend on p'. It follows from Eqs. (2)–(4) that

$$\psi_0 = S_\mu c(\mu) \phi_{0\mu}. \tag{5}$$

One may consider Eq. (5) as a possible concrete definition of the vector ψ_0.

Let us determine the eigenvalue $E_{p'}$ of $\hat{H}$ which corresponds to the common eigenvector $\Phi_{\vec{p}'\mu}$ of $\hat{H}$ and $\hat{\mathbf{P}}$, $\mathbf{p}' \neq 0$. ($E_{p'}$ assumes the value μ if $\mathbf{p}' = 0$). In relativistic theory $E_{p'}^2 - p'^2$ is the Lorentz invariant. From

$$E_{p'}^2 - p'^2 = E_0^2 = \mu^2 \tag{6}$$

the known relativistic formula $E_{p'} = \sqrt{p'^2 + \mu^2}$ follows. So we have

$$\hat{H} \Phi_{p\mu} = \sqrt{p^2 + \mu^2} \Phi_{p\mu} \tag{7}$$

(the prime over p is omitted). It follows from Eqs. (4) and (7) that

$$\Psi_p(t) = e^{-iHt} \Psi_p = S_\mu c(\mu) \Psi_{p\mu} \exp(-it\sqrt{p^2 + \mu^2}). \tag{8}$$

So the probability amplitude of the nondecay (survival amplitude) is

$$\mathcal{A}_p(t) \equiv \langle \Psi_p, e^{-iHt} \Psi_p \rangle = S_\mu |c(\mu)|^2 \exp(-it\sqrt{p^2 + \mu^2}). \tag{9}$$

The state Ψ_p is called unstable if $\mathcal{A}_p(t) \to 0$ as $t \to \infty$. This property holds only if S_μ in Eq. (9) is integral over continual μ values. Further, the spectrum of the Hamiltonian H must be bounded from below. Let us choose the energy origin so that the lower bound in S_μ would be zero. Thus S_μ in Eq. (9) must denote the integral over μ from zero to, e.g., infinity.

When $\mathbf{p} = 0$ Eq. (8) turns into the known equation for the probability amplitude of nondecay of the unstable system at rest, e.g., see Fonda *et al.* (1978).

Equation (8) was obtained by Stefanovich (1996) in a different way. He defined Ψ_p as $\mathcal{U}_p \psi_0$ where $\mathcal{U}_p$ represents Lorentz transformation from the rest frame of unstable particle to the frame where its momentum is $\mathbf{p}$. Here Ψ_p is defined by Eq. (2) and only one relativistic formula, Eq. (6), is used.

3. DILATION OF MOVING UNSTABLE PARTICLE DECAY

Let us calculate and compare the nondecay law $\mathcal{A}_p(t)$ of moving unstable particle and nondecay law $\mathcal{A}_0(t)$ of the particle at rest. For this purpose $|c(\mu)|^2$ is needed, see Eq. (9). It is possible to calculate $|c(\mu)|^2$ using solvable models, (Alzetta and d'Ambrogio, 1966; Gi-Chol Cho *et al.*, 1993; Horwitz, 1995; Levy, 1959). Here the known simplified representation is used (Stefanovich, 1996)

$$|c(\mu)|^2 = \frac{\Gamma}{2\pi}[(\mu - m)^2 + \Gamma^2/4]^{-1}. \tag{10}$$

In general Γ and m are functions of μ, see (Goldberger and Watson, 1964; Messiah, 1961). Here Γ and m are supposed to be constants such that $\Gamma/m \ll 1$ (this is true for all observable unstable systems). I hope that using Eq. (10) allows us to obtain a faithful qualitative notion on the relation of $\mathcal{A}_p(t)$ and $\mathcal{A}_0(t)$.

1. In the Appendix, the main contributions to $\mathcal{A}_p(t)$ and $\mathcal{A}_0(t)$ are calculated using Eq. (10) for all times except extremely small ones $t < 1/m$.

$$\begin{aligned} \mathcal{A}_0(t) &= \frac{\Gamma}{2\pi}\int_0^\infty d\mu[(\mu - m)^2 + \Gamma^2/4]^{-1}\exp(-it\mu) \\ &\cong e^{-itm}e^{-\Gamma t/2} - \frac{i}{m\pi}\frac{\Gamma}{m}\frac{1}{mt}, \quad tm \gg 1, \end{aligned} \tag{11}$$

$$\begin{aligned} \mathcal{A}_p(t) &= \frac{\Gamma}{2\pi}\int_0^\infty d\mu[(\mu - m)^2 + \Gamma^2/4]^{-1}\exp(-it\sqrt{p^2 + \mu^2}) \\ &\cong \exp[-i\sqrt{p^2 + m^2}(1 - \alpha)]\exp[-\Gamma t(1 + \alpha)/2\gamma] \\ &\quad -\frac{1}{2}(2\pi)^{-1/2}\frac{\Gamma}{m}\frac{p}{m}(pt)^{-1/2}\exp(-ipt + i\pi/4). \end{aligned} \tag{12}$$

Equation (12) is true if $t \gg 1/m$ and $pt \gg 1$. The latter condition does not make it possible to get Eq. (11) from Eq. (12) by putting $p = 0$;

$$\gamma = \sqrt{p^2 + m^2}/m = (1 - v^2/c^2)^{-1/2},$$
$$\alpha = \frac{1}{8}\frac{\Gamma^2}{p^2 + m^2}\frac{p^2}{p^2 + m^2} = \frac{1}{8}\frac{\Gamma^2}{m^2}\frac{\gamma^2 - 1}{\gamma^4}. \tag{13}$$

Equations (11) and (12) contain terms proportional to $\exp(-\Gamma t/2)$ and $\exp(-\Gamma t(1+\alpha)/2\gamma)$. Let us call them exponential terms (ET).

Besides ET the equations include terms containing small factors Γ/m and proportional to the inverse powers of t. They are small as compared to ET during several decades of lifetimes. However, they dominate at sufficiently large t. For their discussion see, e.g., Fonda *et al.* (1978), Norman *et al.* (1988), and references therein.

2. The exponential term in Eq. (12) is close to $\exp(-itm\gamma - \Gamma t/2\gamma)$ because $\alpha \ll 1$. Therefore, $|\mathcal{A}_p(t)|^2$ almost coincides with $|\mathcal{A}_0(t/\gamma)|^2$ at the times t when ETs dominate. So, ED approximately holds at these times. Let us estimate corrections to the dilation. One obtains using Eqs. (11) and (12):

$$\{|\mathcal{A}_p(t)|^2 - |\mathcal{A}_0(t/\gamma)|^2\}/|\mathcal{A}_0(t/\gamma)|^2 \cong \Gamma t\alpha/\gamma. \tag{14}$$

Here the difference in the curly brackets decreases when t increases because both minuend and subtrahend decrease as $\exp(-t/\gamma)$. Therefore, the deviation of $|\mathcal{A}_p(t)|^2$ from $|\mathcal{A}_0(t/\gamma)|^2$ should be characterized by the ratio of this difference to $|\mathcal{A}_0(t/\gamma)|^2$. The ratio is equal to $\Gamma t\alpha/\gamma$ and grows as t increases. However, it is extremely small even if, e.g., $\Gamma t \sim 100$ because of $\alpha \ll 1$. One cannot consider still greater times because then nonexponential terms will dominate in Eqs. (11) and (12). It follows from Eqs. (14) and (13) that the maximal deviation occurs when $\gamma^2 = 5/3$.

The estimation shows that measuring deviation from ED in the exponential time region needs much more accuracy than is achieved in existing experiments (which is about $0.1 - 0.2\%$) (Bailey *et al.*, 1977; Farley, 1992).

The same conclusion was obtained by Stefanovich (1996) by means of numerical calculations of the difference $|\mathcal{A}_p(t)|^2 - |\mathcal{A}_0(t/\gamma)|^2$ for $\Gamma/m = 2 \cdot 10^{-4}$ and several values of t and γ. Note that my analytical estimate gives much smaller values for the difference, as compared to those presented in Fig. 1 of Stefanovich's paper.

3. ED decisively fails for asymptotically large times when nonexponential terms in $|\mathcal{A}_p(t)|^2$ and $|\mathcal{A}_0(t)|^2$ dominate. Indeed, then $|\mathcal{A}_p(t)|^2 \sim t^{-1}$ while $|\mathcal{A}_0(t)|^2 \sim t^{-2}$, see Eqs. (11) and (12). So, the former is delayed as compared to the latter but this is not ED $t \to t/\gamma$.

For the same reason ED fails also in the transient (preasymptotical) region of times when exponential terms are of the same order as the asymptotical ones.

It is appropriate to note here that the asymptotic behavior of the nondecay amplitude is sensitive to the $\Gamma(\mu)$ behavior at small μ (remind that Γ is here supposed to be a constant). It is likely that $\mathcal{A}_p(t)$ and $\mathcal{A}_0(t)$ have different asymptotic inverse power behaviors at any $\Gamma(\mu)$.

However, when asymptotic terms begin to dominate the decay is almost completed and we have nothing to observe. So far experimenters fail to measure nonexponential asymptotics of the decay law (Norman *et al.*, 1988 and references therein).

In the next section, I consider another variant of the quantum clock in which ED fails for all times under some condition.

4. Exner (1983) claimed the validity of ED for the decay law of moving unstable particles (see his section V). Let us comment on his approach.

Instead of the initial state Ψ_p with definite momentum Exner considered a packet with almost definite momentum. In this case, the definition (9) of the survival amplitude is not suitable. Indeed, then the survival amplitude changes with time not only because of the decay but also because of the packet diffusion. Besides, the corresponding amplitude of moving particles changes due to the initial packet displacement at the vector $\mathbf{v}t$, $\mathbf{v}$ being the packet average velocity (under the displacement our Ψ_p does not change up to an unessential phase factor). So, in the case one must use a more general definition of the survival law (e.g., see Eq. (3a) in Exner, 1983). Nevertheless, Exner used definition (9) which in his approach may be considered as a simplified approximate definition of the nondecay law. He stipulated that his claim on the ED validity holds provided this approximation is admissible. In particular, he made the reservation that the decay law is considered in the time region where the decay law is exponential. Exner did not estimate the validity of his approximation. A refinement of Exner's approach may result in ED violation, although small. This would agree with the conclusion obtained in Subsection 3.2: ED holds only approximately when exponential terms dominate.

4. K_0-MESON-LIKE SYSTEMS AND EINSTEIN DILATION

Consider two unstable particles K_s and K_l with different masses and lifetimes, i.e., different distributions $|c_s(\mu)|^2$ and $|c_l(\mu)|^2$, see Eq. (10). I call the particles K_s and K_l because the known mesons K_s and K_l may be familiar examples. However, here their masses m_s and m_l are not supposed to be close. As usual, let us suppose that $\Gamma_s/m_s \ll 1$ and $\Gamma_l/m_l \ll 1$. Let $K_s(t)$ and $K_l(t)$ denote the corresponding survival amplitudes of the particles at rest, see Eq. (11) where Γ and m are replaced by Γ_s, m_s, and Γ_l, m_l, respectively.

Let us suppose that at $t = 0$ one may prepare the state

$$[|K_s\rangle + |K_l\rangle]/\sqrt{2} \equiv |K_0\rangle. \tag{15}$$

For example, in the case of real K_0 meson the state $|K_0\rangle$ is the product of the reaction $\pi^- + p \to K_0 + \Lambda_0$. Consider the survival amplitude of the state

$$\begin{aligned} K(t) &= \frac{1}{2}([|K_s\rangle + |K_l\rangle], \exp(-iHt)[|K_s\rangle + |K_l\rangle]) \\ &= \frac{1}{2}(|K_s\rangle, \exp(-iHt)|K_s\rangle) + \frac{1}{2}|K_l\rangle, \exp(-iHt)|K_l\rangle) \\ &= \frac{1}{2}K_s(t) + \frac{1}{2}K_l(t). \end{aligned} \tag{16}$$

I supposed above that the vector $|K_s\rangle$ (which describes the particle K_s) as well as all vectors of the K_s decay products are orthogonal to $|K_l\rangle$ and to its decay product vectors. This supposition is valid for the real mesons K_s and K_l because K_s and K_l and their decay products have different CP-parities (CP conservation holds almost exactly). Neglecting nonexponential terms in Eq. (11) for $K_s(t)$ and $K_s(t)$ one obtains for the K_0 survival probability $|K(t)|^2$

$$|K(t)|^2 = \frac{1}{4}[e^{-t\Gamma_s} + e^{-t\Gamma_l} + 2\,\exp(-t(\Gamma_s + \Gamma_l)/2)\cos(m_s - m_s)t] \tag{17}$$

Cf. equation (7.83) in Perkins (1987). Equation (17) may be represented in the form

$$|K(t)|^2 = \exp(-t(\Gamma_s + \Gamma_l)/2)\cos^2 t(m_s - m_s)/2 + \sinh^2 t(\Gamma_s - \Gamma_l)/4] \tag{18}$$

which presents more visually oscillations which $|K(t)|^2$ has. For example, when $t\Gamma_s \ll 1$ and $t\Gamma_l \ll 1$ we have $|K(t)|^2 \cong \cos^2 t(m_s - m_l)/2$.

Analogously, consider the moving K_0 system having nonzero definite $\mathbf{p}$. The related survival amplitude $K_p(t)$ is a superposition $[K_{sp}(t) + K_{lp}(t)]/\sqrt{2}$, where $K_{sp}(t)$ and $K_{lp}(t)$ are defined by equations of the kind of Eq. (12) (Γ and m being replaced by Γ_s, m_s, and Γ_l, m_s). As above, only exponential terms are retained. The small corrections α_s and α_l are also neglected. Thus,

$$K_{sp}(t) \cong \exp\left(-it\sqrt{p^2 + m_s^2}\right)\,\exp(-t\Gamma_s/2\gamma_s),\ \gamma_s \equiv \sqrt{p^2 + m_s^2}/m_s, \tag{19}$$

$$K_{lp}(t) \cong \exp\left(-it\sqrt{p^2 + m_l^2}\right)\,\exp(-t\Gamma_l/2\gamma_l),\ \gamma_l \equiv \sqrt{p^2 + m_l^2}/m_l, \tag{20}$$

$$\begin{aligned} |K_p(t)|^2 &= \frac{1}{2}|K_{sp}(t) + K_{lp}(t)|^2 \\ &= \exp\left(-\frac{t}{2}\left(\frac{\Gamma_s}{\gamma_s} + \frac{\Gamma_l}{\gamma_l}\right)\right)\left[\cos^2\left(\frac{t}{2}\left(\sqrt{p^2 + m_s^2} - \sqrt{p^2 + m_l^2}\right)\right)\right. \\ &\quad \left. + \sinh^2\frac{t}{4}\left(\frac{\Gamma_s}{\gamma_s} - \frac{\Gamma_l}{\gamma_l}\right)\right]. \end{aligned} \tag{21}$$

Note. The oscillations which are present in $K(t)$ and $K_p(t)$ allow us to use the unstable system described by the vector $|K_0\rangle = [|K_s\rangle + |K_l\rangle]/\sqrt{2}$ as a quantum clock in a different way as compared with the usual unstable system (see Introduction section). Namely, the unit of time provided by $|K(t)|^2$ may be defined as the period $(m_s - m_s)^{-1}$ of $|K(t)|^2$ oscillations, $m_s - m_l$ being the oscillation frequency. Analogously, the moving K_0 determines the unit of time Δ_p^{-1}, $\Delta_p = \sqrt{p^2 + m_s^2} - \sqrt{p^2 + m_s^2}$ being the oscillation frequency of $|K_p(t)|^2$, see Eq. (21).

We can see from Eqs. (18) and (21) that $|K_p(t)|^2$ can be obtained from $|K(t)|^2$ by the replacements

$$\Gamma_s \to \Gamma_s/\gamma_s, \quad \Gamma_l \to \Gamma_l/\gamma_l, \quad \Delta m \to \Delta_p, \tag{22}$$

$$\Delta m \equiv m_s - m_l, \quad \Delta_p \equiv \sqrt{p^2 + m_s^2} - \sqrt{p^2 + m_l^2}. \tag{23}$$

Multiplying and dividing Δ_p by $\sqrt{p^2 + m_s^2} + \sqrt{p^2 + m_l^2}$ one obtains for Δ_p

$$\Delta_p = \left(m_s^2 - m_l^2\right)\left[\sqrt{p^2 + m_s^2} + \sqrt{p^2 + m_l^2}\right]^{-1} = \Delta m/\tilde{\gamma},$$
$$\tilde{\gamma} = \left[\sqrt{p^2 + m_s^2} + \sqrt{p^2 + m_l^2}\right]/(m_s + m_l). \tag{24}$$

So the last replacement in (22) may be represented as $\Delta m \to \Delta m/\tilde{\gamma}$. The replacements (22) cannot be reduced to one replacement $t \to t/\gamma$ if $\gamma_s \neq \gamma_l$. So ED decisively fails in this case. The inequality $\gamma_s \neq \gamma_l$ holds if $m_s \neq m_l$ (provided p is not too small: if $p \ll m_s, m_l$ then $\gamma_s \cong \gamma_l \cong \tilde{\gamma} \cong 1$).

ED holds approximately if $m_s \cong m_l$ (this is the case for real K_s and K_l). Then $\gamma_s \cong \gamma_l \cong \tilde{\gamma}$ and (22) may be reduced to $t \to t/\gamma$.

5. CONCLUSION

Evaluations of the decay law of moving unstable particles show that Einstein time dilation $t \to t/\gamma$ (ED) is not the exact kinematical law of relativistic quantum mechanics. The deviation from ED is small for usual unstable systems when the time t of the decay does not exceed several decades of lifetimes. The estimation of the value of the deviation is obtained in Section 3, see Eqs. (14) and (13). It linearly grows as t increases (in the above region) and is maximal at $\gamma^2 = 5/3$. It is much less than the achieved experimental accuracy. It was shown that ED does not hold decisively for larger times when the decay is nonexponential. ED fails for all times in the case of unstable system of the kind $K_0 = (K_s + K_l)/\sqrt{2}$ with appreciably different masses m_s and m_l. More explicitly, in both two latter cases we have dilations, see Subsection 3.3 and Eq. (22) in Section 4, but the dilations are not Einsteinian. However, experimenters fail to observe nonexponential decay while K_0-meson-like system with $m_s \neq m_l$ are unknown. Thus, for the unstable decays available now ED turns out to hold approximately with great accuracy determined here by analytical calculations.

APPENDIX: EVALUATION OF SURVIVAL AMPLITUDE

The survival amplitude of the moving unstable particle follows from Eqs. (9) and (10)

$$\mathcal{A}_p(t) = \int_0^\infty dx w(x) \exp\left(-it\sqrt{p^2 + x^2}\right), \tag{A1}$$

$$w(x) = \frac{\Gamma}{2\pi}[(x - m)^2 + \Gamma^2/4]^{-1} \tag{A2}$$

(μ is replaced by x). To calculate (A1) the integration contour C is chosen, see Fig. A1. The integrand in (A1) is the single-valued analytical function inside C (with the pole at the point $m - i\Gamma/2$). Its branch is chosen which is positive on the real positive semiaxis x: $\arg\sqrt{z^2 + p^2} = 0$ when $z = x$. Then on the quarter of the circle C_R the argument of $\sqrt{z^2 + p^2}$ coincides with arg z (the radius R of C_R being large) and is zero at the point $x = R$, $y = 0$, and is equal to $-\pi/2$ at the point $x = 0$, $y = iR$. When z moves along C_i from iR up to the branch point $-ip$ the arg $\sqrt{z^2 + p^2}$ does not change remaining to be equal to $-\pi/2$. This follows from the equation

$$\arg\sqrt{z^2 + p^2} = \frac{1}{2}[\arg(z - ip) + \arg(z + ip)]. \tag{A3}$$

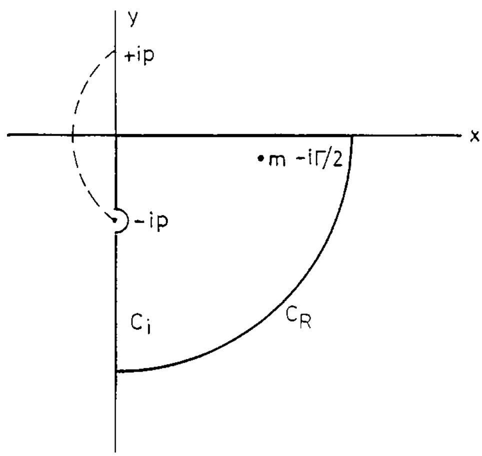

Fig. A1. Integration contour for $\mathcal{A}_p(t)$. Dotted line shows the chosen branch cut for the function $\sqrt{z^2 + p^2}$.

So on the interval $(-iR, -ip)$ we have $\sqrt{z^2+p^2} = -i|\sqrt{y^2-p^2}|$. Further as z passes the point $(-ip)$ the argument of $(z-ip)$ does not change as before while $\arg(z+ip)$ changes by $+\pi$ in the vicinity of $(-ip)$. Therefore, $\arg\sqrt{z^2+p^2}$ changes by $\pi/2$, see Eq. (A3), and $\sqrt{z^2+p^2}$ becomes positive real being equal to $|\sqrt{p^2-y^2}|$. The above consideration of the behavior of the chosen branch of $\sqrt{z^2+p^2}$ gives

$$\int_C dzw(z)\exp\big(-it\sqrt{z^2+p^2}\big) = \int_0^\infty dxw(x)\exp\big(-it\sqrt{x^2+p^2}\big)$$
$$+\int_R^p (-idy)w(-iy)\exp\big(-t\big|\sqrt{y^2-p^2}\big|\big)$$
$$+\int_p^0 (-idy)w(-iy)\exp\big(-it\big|\sqrt{p^2-y^2}\big|\big) = -2\pi i\text{Res}. \tag{A4}$$

Here Res is the residue of the integrand at the pole $m - i\Gamma/2$:

$$\text{Res} = (-2\pi i)^{-1}\exp\big[-it\sqrt{(m-i\Gamma/2)^2+p^2}\big]. \tag{A5}$$

The integral over C_R is not written in Eq. (A4) because it vanishes as $R\to\infty$ due to the Jordan lemma.

To calculate $\mathcal{A}_p(t)$, see Eq. (A1) and (A4), we must now compute the integrals

$$I_{\infty p} = i\int_p^\infty dy\frac{\Gamma}{2\pi}[(iy+m)^2+\Gamma^2/4]^{-1}\exp\big(-t\big|\sqrt{y^2-p^2}\big|\big), \tag{A6}$$

$$I_{p0} = i\int_0^p dy\frac{\Gamma}{2\pi}[(iy+m)^2+\Gamma^2/4]^{-1}\exp\big(-it\big|\sqrt{p^2-y^2}\big|\big). \tag{A7}$$

Let us estimate $I_{\infty p}$. The main contribution to $I_{\infty p}$ is due to y values close to p, if t is not too small. One may suppose, e.g., that $t \gg 1/m$ or $t > 0.1\tau = 0.1/\Gamma$. In order to obtain an approximate value for $I_{\infty p}$, let us replace $(iy+m)^2+\Gamma^2/4$ in the integrand by $(ip+m)^2$. Then

$$I_{\infty p} \cong \frac{i\Gamma}{2\pi}(ip+m)^{-2}J_{\infty p}, \quad J_{\infty p} = \int_p^\infty dy\exp\big(-t\sqrt{y^2-p^2}\big). \tag{A8}$$

After the change of variables $u = \sqrt{y^2-p^2}$ the integral $I_{\infty p}$ reduces to the table one

$$J_{\infty p} = \int_0^\infty du\frac{u}{\sqrt{u^2+p^2}}e^{-tu}, \tag{A9}$$

(e.g., see 3.366.3 in Gradshtein and Ryzhik (1962). I shall need the value of $J_{\infty p}$ at $pt \gg 1$ (this means that p is supposed to be not too small). Using formulae 3.366.3 and 8.554 in the cited tables (or immediately by integrating (A9) by parts) we get

$$J_{\infty p} \cong p(pt)^{-2} \tag{A10}$$

$$|I_{\infty p}| = \frac{1}{2\pi}\frac{\Gamma}{\sqrt{p^2+m^2}}\frac{1}{\sqrt{p^2+m^2t}}\frac{1}{pt}, \quad mt \gg 1, \quad pt \gg 1. \tag{A11}$$

Now let us estimate I_{p0}, see Eq. (A7). After the change of variables $v = \sqrt{p^2 - y^2}$ we get (neglecting $\Gamma^2/4$)

$$I_{p0} = \frac{i\Gamma}{2\pi}\int_0^p dv \frac{v}{\sqrt{p^2-v^2}}\left[i\sqrt{p^2-v^2}+m\right]^{-2}\exp(-itv). \tag{A12}$$

The values of v close to p bring the main contribution due to the factor $(p^2 - v^2)^{-1/2}$. So,

$$I_{p0} \cong \frac{i\Gamma}{2\pi m^2}\left\{\int_0^p dv v\frac{\cos tv}{\sqrt{p^2-v^2}} - i\int_0^p dv v\frac{\sin tv}{\sqrt{p^2-v^2}}\right\}$$

$$\equiv \frac{i\Gamma}{2\pi m^2}\{I_C + iI_S\}. \tag{A13}$$

For the integrals I_C and I_S see the formula 2.5.8.2 (with $\beta = 1/2$) in Prudnikov (2003) (note that in the first edition of the book there is a misprint in this formula)

$$I_S = \frac{\pi}{2}pJ_1(pt), \quad I_C = -\frac{\pi}{2}p\mathbf{H}_1(pt) + p.$$

Here J_1 is the Bessel function and $\mathbf{H}_1$ is the Struve function. Using their asymptotic expansions (see, e.g., 8.451.1 and 8.554, 8.451.2 in Gradshtein and Ryzhik (1962)) one gets at $pt \gg 1$

$$I_S \cong -\sqrt{\frac{\pi}{2}}\frac{p}{\sqrt{pt}}\cos(pt+\pi/4), \quad \cong I_C\sqrt{\frac{\pi}{2}}\frac{p}{\sqrt{pt}}\sin(pt+\pi/4), \tag{A14}$$

$$I_{p0} \cong \frac{1}{2\pi}(2\pi)^{-1/2}\frac{\Gamma}{m}\sqrt{\frac{p}{m}}\frac{1}{\sqrt{mt}}\exp(-ipt+i\pi/4), \quad pt \gg 1.$$

We see that the integral $I_{\infty 0}$, Eq. (A11) can be neglected as compared to I_{p0} in the case $pt \gg 1$. Therefore, $I_{\infty p}$ is omitted in Eq. (12) for $\mathcal{A}_p(t)$.

One can see that Res, Eq. (A5), dominates in $\mathcal{A}_p(t)$ during several decades of lifetimes while I_{p0} gives $\mathcal{A}_p(t)$ asymptotics as $t \to 0$.

The amplitude $\mathcal{A}_0(t)$ is calculated in the same way, the result being given by Eq. (11). Our goal is the investigation of the dilation of $\mathcal{A}_p(t)$ as compared to $\mathcal{A}_0(t)$. For this purpose, one has first of all to compare Res in $\mathcal{A}_p(t)$, see Eq. (A5), with Res in $\mathcal{A}_0(t)$, see the first term in Eq. (11). Let us represent Res, Eq. (A5), in the form best suitable for this comparison (see Subsection 3.2). As $\Gamma/m \ll 1$ one may expand

$$[(m - i\Gamma/2)^2 + p^2]^{1/2} = \sqrt{p^2 + m^2}[1 - (i\Gamma m + \Gamma^2/4)(p^2 + m^2)^{-1}]^{1/2}$$

in the series over powers of Γ. Omitting the terms $\sim \Gamma^4$ and still lesser ones we obtain

$$[(m - i\Gamma/2)^2 + p^2]^{1/2} = \sqrt{p^2 + m^2}(1 - \alpha) - i\Gamma(1 + \alpha)/2\gamma + \cdots, \qquad \text{(A15)}$$

for α and γ see Eq. (13). Eq. (12) follows from Eqs. (A1), (A4), (A5), (A14), and (A15).

Note. Instead of expansion (A15) Stefanovich (1996) used the expansion of the square root $\sqrt{p^2 + x^2}$ in the integrand of (A1) in series over powers of the ratio $(x - m)[p^2 + m^2]^{-1/2}$. He made the implication that x values close to m predominate in the integrand because $w(x)$ is peaked when $-\Gamma < x - m < \Gamma$, $\Gamma/m \ll 1$. Then only the first terms of the series dominate and $|\mathcal{A}_p(t)|^2$ turns out to be approximately equal to $|\mathcal{A}_p(t/\gamma)|^2$ notwithstanding t values, i.e., ED results for all t. However, my analytical calculation of $\mathcal{A}_p(t)$ shows that ED decisively fails at large t when the decay is not exponential. So Stefanovich's implication actually is false for large t (e.g., because x values from the interval $(m - \Gamma, m + \Gamma)$ are suppressed by fast oscillations of $\exp(-it\sqrt{p^2 + x^2})$ at large t).

Let us stress that expansion (A15) (unlike Stefanovich's one) is done after analytical evaluation of integral (A1). The validity of (A15) does not depend on Stefanovich's implication, only the smallness of Γ/m being essential.

Note also that Stefanovich's expansion does not give an explicit estimation of the ED violation (which is characterized here by the quantity α, see Eq. (15)). For this estimation he numerically calculated the involved integrals (for times not exceeding 10 lifetimes, see his Fig. 1).

REFERENCES

Alzetta, A. and d'Ambrogio, E. (1966). Evolution of a resonant state. *Nuclear Physics* **82**, 683–689.

Bailey, J., *et al.* (1977). Measurement of relativistic time dilation for positive and negative muons in a circular orbit. *Nature* **268**, 301–304.

Exner, P. (1983). Representations of the Poincaré group associated with unstable particles. *Physical Review D* **28**(10), 2621–2627.

Farley, F. (1992). The CERN (g-2) measurements. *Zeitschrift fur Physik C* **56**, S88, Sect. 5.

Fonda, L., Ghirardi, G., and Rimini, A. (1978). Decay theory of unstable quantum system. *Reports on Progren in Physics* **41**, 587, Sect. 3.

Gi-Chol Cho, *et al.* (1993). The time evolution of unstable particles. *Progress in Theoretical Physics* **90**(4), 803–816.

Goldberger, M. and Watson, K. (1964). *Collision Theory*, Wiley, New York, Chap. 8.

Gradshtein, I. and Ryzhik, I. (1962). *Tables of Integrals, Sums, Series and Products*, GIFML, Moscow.

Horwitz, L. (1995). The unstable system in relativistic quantum mechanics. *Foundation of Physics* **25**(1), 39–65.

Levy, M. (1959). On the description of unstable particles in quantum field theory. *Nuovo Cimento* **13**(1), 115.
Messiah, A. (1961). *Quantum Mechanics, Vol. 2*, North-Holland, Amsterdam, Chap. 21, 13.
Møller, C. (1972). *The Theory of Relativity*, Clarendon Press, Oxford, Chap. 2.6.
Norman, E. *et al*. (1988). Tests of the exponential decay law at short and long times. *Physical Review Letters* **60**(22), 2246.
Perkins, D. (1987). *Introduction to High Energy Physics*, Addison-Wesley Reading, MA.
Prudnikov, A. P., *et al*. (2003). *Integrals and Series, Vol. 1*, 2nd edn., Fizmatlit, Moscow.
Stefanovich, E. (1996). Quantum effects in relativistic decays. *International Journal of Theoretical Physics* **35**(12), 2539–2554.